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THE ADHESION OF TWO ELASTIC BODIES WITH SLIGHTLY WAVY SURFACES

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Abstract—The stresses at the contact of two one-dimensional wavy surfaces in the presence of adhesion between them are found by the combination of the solution in the absence of adhesion due to Westergaard (1939, *J. Appl. Mech. Trans. ASME* **6,** 49–53), with that for an array of collinear cracks due to Koiter (1959, *Ing. Arch.* **28,** 168–172). The mean contact pressure necessary to obtain full contact is found. Separation arises from small flaws at the interface. The relation between the size of the flaw and the mean tension to break the joint is obtained. The corresponding relationship is also found for a surface having orthogonal two-dimensional waves of equal amplitude and wavelength.

1. INTRODUCTION

The analysis in this paper relates to the conditions of adhesion between two elastic bodies whose surfaces, though nominally flat, have a sinusoidal undulation of small amplitude. The bodies are represented by two elastic half-spaces. If the waves on each contacting surface are one-dimensional, parallel and of equal wavelength λ, the gap between them prior to deformation can be expressed by

$$h(x) = \Delta\{1-\cos(2\pi x/\lambda)\}, \tag{1}$$

in which $\Delta \ll \lambda$ [see Fig. 1(a)]. When the surfaces are pressed into contact, adhesion between them can arise through the action of molecular forces, if the surfaces are sufficiently clean, or through a thin film of adhesive (glue). Adhesive strength is characterised by the *work of adhesion w*, which is the work required to separate a unit area of adhered interface, and is usually measured by some form of peel test [e.g. Crocombe & Adams (1981)]. Where adhesion is due to surface forces,

$$w = \Delta\gamma = \gamma_1+\gamma_2-\gamma_{12}, \tag{2}$$

where γ_1 and γ_2 are the surface energies of each surface and γ_{12} is the energy of the interface. Adhesion arising in this way has been widely studied and measured, in particular using (i) cleaved mica (Israelachvili, 1985) and (ii) soft, smooth rubber (Roberts, 1975). In the case of an interspersed layer of glue, the thickness of the layer is much less than the amplitude of the wave.

The contact of elastic spheres was investigated theoretically and experimentally by Johnson *et al.* (1971) (JKR theory). It was shown that the area of contact under a compressive load exceeded that given by the Hertz theory, that the spheres remained in contact over a finite area when the load was removed, and that it required a tensile force to pull the spheres apart, given by

$$P_c = (3/2)\pi w R, \tag{3}$$

where R is related to the radii $R_{1,2}$ of the spheres by $R = R_1R_2/(R_1+R_2)$.

The JKR theory was developed through a balance of changes in surface energy and elastic strain energy, as in the Griffith's theory of brittle fracture. Later Maugis and Barquins

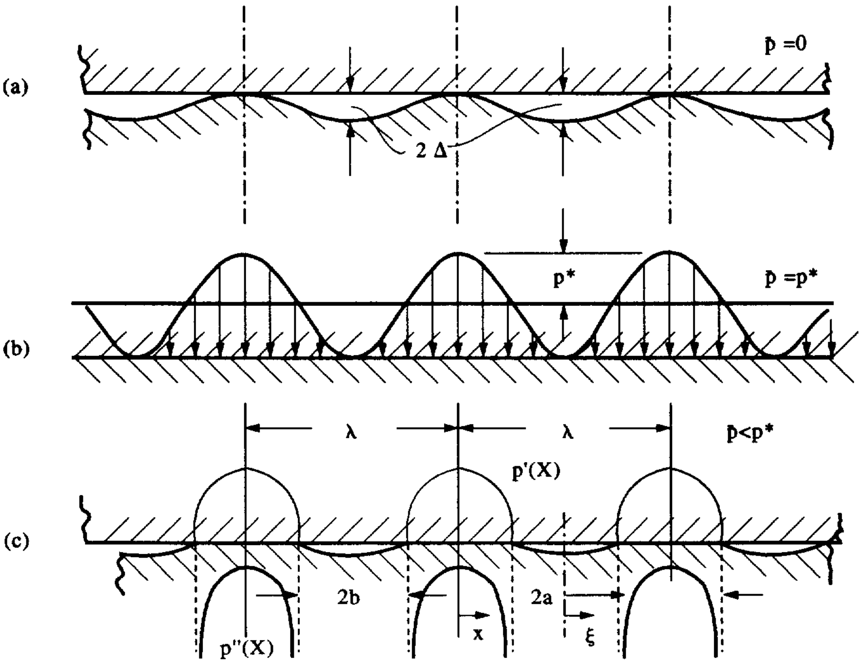

Fig. 1. Contact of a flat surface with an elastic wavy surface: (a) undeformed gap $h(x)$ expressed by eqn (1); (b) complete contact, $\bar{p} = p^*$; (c) partial contact on strips of width $2a$. $p'(x)$ = pressure distribution without adhesion; $p''(x)$ = stress distribution for an array of cracks of length $2b$; $p(x) = p'(x)+p''(x)$ = surface traction with adhesion.

(1978) showed how the same results could be obtained more directly by the application of linear elastic fracture mechanics. This is the approach which will be followed here.

2. ANALYSIS

The contact of two slightly wavy half-planes in the absence of adhesion was analysed first by Westergaard (1939), using the stress function which carries his name, and later by Dundurs *et al.* (1973) by reducing the problem to an Abel integral equation. The half-planes have Young's moduli $E_{1,2}$ and Poisson's ratios $\nu_{1,2}$. We write $1/E^* = [(1-\nu_1^2)/E^1 + (1-\nu_2^2)/E_2]$, and the interface is assumed to be frictionless.

If the pressure $\bar{p}'$, averaged over the whole surface, exceeds the value $p^* = \pi E^*\Delta/\lambda$, contact will be made throughout the whole interface [Fig. 1 (b)]. If $\bar{p}' < p^*$, contact occurs on strips of width $2a$ located at the crests of the waves, given by

$$(\pi a/\lambda) = \sin^{-1}(\bar{p}'/p^*)^{1/2}. \tag{4}$$

The pressure distribution at each contact $(-a < x < a)$ is given by

$$p'(x) = \frac{2\bar{p}'\cos\psi}{\sin^2\psi_a}(\sin^2\psi_a - \sin^2\psi) \tag{5}$$

where $\psi = \pi x/\lambda$ and $\psi_a = \pi a/\lambda$, [see Fig. 1(c)].

We now imagine that the mean pressure is reduced from $\bar{p}'$ to $\bar{p}$ by the superposition of a negative (tensile) pressure $\bar{p}''$, during which the surfaces in the contact area $(-a < x < a)$ remain adhered together. It will be readily appreciated from Fig. 1(c) that this step corresponds to the tensile loading of a plane which contains an array of equally spaced cracks, each of length $2b$ $(= \lambda - 2a)$. This problem has been analysed by Koiter (1959). The stress across each ligament $(-b < x < b)$ is given by

$$p''(\xi) = -\bar{p}''[1-\{\sin(\pi b/\lambda)/\sin(\pi\xi/\lambda)\}^2]^{-1/2}$$

i.e.

$$p''(x) = -\bar{p}''(1-\{\cos\psi_a/\cos\psi\}^2]^{-1/2}. \tag{6}$$

This stress distribution gives rise to a mode I stress intensity factor at the edges of the contact:

$$\begin{aligned} K_{\rm I} &= -\bar{p}''\{\lambda\tan(\pi b/\lambda)\}^{1/2} \\ &= -\bar{p}''\{\lambda\cos\psi_a\}^{1/2}. \end{aligned} \tag{7}$$

The net pressure distribution at the contacts is then given by the superposition of $p'(x)$ and $p''(x)$. It is compressive in the centre ($x = 0$) and has tensile singularities at the edges ($x = a$).

We can now determine the equilibrium value of a by equating the elastic strain energy release rate G to the work of adhesion w, whereby

$$G = \frac{k_{\rm I}^2}{2E^*} = w$$

i.e.

$$\bar{p}''^2\frac{\lambda\cot\psi_a}{2E^*} = w.$$

Then substituting in eqn (7) gives

$$\begin{aligned} \frac{\bar{p}''}{p^*} &= -\left[\frac{2E^*w}{\lambda p^{*2}}\tan\psi_a\right]^{1/2} \\ &= -\alpha\{\tan\psi_a\}^{1/2} \end{aligned} \tag{8}$$

where

$$\alpha \equiv \left\{\frac{2E^*w}{\lambda p^{*2}}\right\}^{1/2} = \left\{\frac{2\lambda w}{\pi^2\Delta^2E^*}\right\}^{1/2}.$$

The net mean pressure $\bar{p}$ is related to the equilibrium contact size a by

$$\frac{\bar{p}}{p^*} = \frac{\bar{p}'}{p^*} + \frac{\bar{p}''}{p^*}$$

i.e.

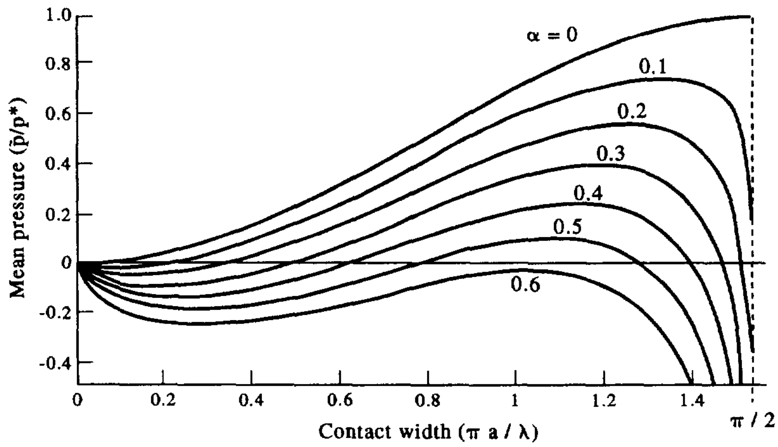

Fig. 2. Partial contact: variation of semi-contact width a (normalised by wavelength λ) with mean pressure (normalised by p^*).

$$\bar{p}/p^* = \sin^2\psi_a - \alpha\{\tan\psi_a\}^{1/2}. \tag{9}$$

This relationship is plotted in Fig. 2 for a series of values of α.

When adhesion is absent, $\alpha = 0$ and eqn (9) reduces to eqn (4). When ψ_a is small, so that contact is confined to narrow strips close to the crest of each wave, eqn (9) reduces to

$$\lambda\bar{p} = \frac{\pi^3 \Delta E^* a^2}{\lambda^2} - \{2\pi E^* wa\}^{1/2}. \tag{10}$$

The load per unit length of each ridge $P = \lambda\bar{p}$ and the radius of curvature of a crest $R = \lambda^2/4\pi^2\Delta$, whereupon eqn (10) may be written as

$$P = \frac{\pi E^* a^2}{4R} - (2\pi E^* wa)^{1/2}. \tag{11}$$

This is the relationship for the contact of two elastic cylinders with parallel axes in the presence of adhesion.

3. INTERPRETATION

Since the solids are elastic and surface forces are conservative, bringing the surfaces together and separating them under *equilibrium conditions*, i.e. following one of the curves in Fig. 2, is a reversible process. However this is not what normally happens in practice, as we shall see in the following discussion. The non-dimensional parameter α^2 represents the ratio of the surface energy in one wavelength to the elastic strain energy when the wave is flattened. The curve of $(\bar{p}/p^*)$ against $(\pi a/\lambda)$, for $\alpha = 0.3$, is reproduced in Fig. 3. It has zero-crossings at A and B, a maximum (pressure) at C and a minimum (i.e. maximum tension) at D. The value of w for metals is about 1.0 J/m^2 and that for rubber about 0.03 J/m^2; corresponding values of E^* are 3×10^{10} and 10^5 N/m^2. Taking a representative wavelength $\lambda = 10$ mm, a value of $\alpha = 0.03$ is obtained by an amplitude $\Delta \simeq 3$ μm for metals and $\simeq 80$ μm for rubber.

If two such surfaces are brought into contact under zero load, they will immediately snap together under the action of surface forces until equilibrium is reached at point B in Fig. 3. This phase of the process is irreversible and takes place at a speed approaching the Rayleigh wave speed. The work done by the surface forces (i.e. the reduction in surface energy) exceeds the increase in elastic strain energy associated with the deformation at point B, the excess being dissipated in the radiation of stress waves.

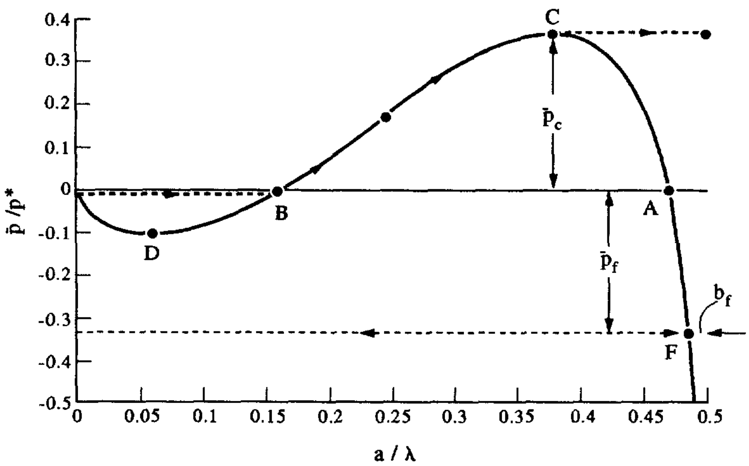

Fig. 3. Contact of elastic wavy surfaces with adhesion: compression–tension cycle. A mean pressure $\bar{p}_c$ is required to make full contact. A tensile stress of $(p^* - \bar{p}_f)$ is required to propagate a flaw of width $2b_f$).

To increase the contact area further, a compressive load (positive $\bar{p}$) is required which, if applied slowly, follows the equilibrium curve from B to C. This point is also unstable; beyond it the surfaces snap into complete contact at $a = \lambda/2$, and remain so when the load is removed.

To separate the surfaces a tensile force (negative $\bar{p}$) is required. The singularity in the equilibrium curve implies that a perfect joint exhibits the theoretical strength of the interface, which is much larger than is generally observed in practice. To obtain realistic values for the strength, as in fracture mechanics of brittle materials, interfacial flaws of finite size must be invoked. In the present situation they could arise from trapped air, contaminants, or fine scale roughness of the surfaces.

We shall start by assuming that a flaw of width $2b_f$, in which no surface forces act, is located at a trough $(x = \lambda/2)$. The condition of zero surface forces within the flaw can be thought of as the superposition of the pressure necessary to keep the surfaces in contact, given by

$$p(\xi) = p^*\{1 - \cos(2\pi\xi/\lambda)\}, \tag{12}$$

and an equal negative pressure acting on the surfaces of the flaw $(-b_f < \xi < b_f)$. Provided $b_f \ll \lambda/2$ the pressure within the flaw may be approximated by the parabolic relationship

$$p(\xi) \simeq (p^* - \bar{p}) - 2\pi^2(\xi/\lambda)^2 p^*. \tag{13}$$

These forces give rise to a stress intensity factor at the ends of the flaw, given by

$$K_I = (\pi b_f)^{-1/2} \int_{-b}^{b} p(\xi)\{(b_f + \xi)/(b_f - \xi)\}^{1/2}\, d\xi$$

$$= (\pi b_f)^{-1/2} \pi[(p^* - \bar{p}) - (\pi b_f/\lambda)^2]. \tag{14}$$

The flaw will extend when $K_I^2/wE^* = w$, at a stress $\bar{p}_f$ given by

$$\frac{\bar{p}_f}{p^*} = 1 - \frac{\lambda}{\pi b_f}\left[\alpha + \left(\frac{\pi b_f}{\lambda}\right)^3\right] \tag{15a}$$

$$\simeq 1 - \alpha(\lambda/\pi b_f), \quad (\pi b_f/\lambda) \ll \alpha. \tag{15b}$$

It may easily be shown that eqn (15b) expresses the condition that a plane-strain crack of length $2b_f$ in an infinite solid should propagate under a far-field stress of $(p^* - \bar{p})$. If this critical value of tension (negative $\bar{p}$) is exceeded, the flaw will expand in an unstable way, leading to complete rupture of the joint.

If the approximation in eqn (15b) is maintained (i.e. $\pi b_f/\lambda \ll \alpha$), it is simple to find the critical stress to propagate a small flaw at any location $x = x_f$. At this position the pressure to maintain full contact is given by eqn (12). Hence, for a flaw located at $x = x_f$, eqn (15b) for the critical stress becomes

$$\bar{p}/p^* = -\cos(2\pi x_f/\lambda) - \alpha(\lambda/\pi b_f). \tag{16}$$

For a flaw located at a trough ($x_f = \lambda/2$), eqn (16) reduces to eqn (15b). In any other position the tension required to propagate the flaw increases, being greatest at a crest, where the contact pressure tending to keep the flaw closed is a maximum.

So far the discussion has assumed that the solids are perfectly elastic, but most real materials display some inelasticity at the high local strains which occur at the edge of an adhesive contact. Rubber, in particular, which shows strong contact adhesion, exhibits strong viscoelastic behaviour at the high strain rates which occur when the contact edge is moving. Also most adhesives have viscoelastic properties. If the surfaces are separating, the external force has to overcome viscoelastic losses in addition to surface energy, leading to an apparent work of adhesion w_e which exceeds the intrinsic surface energy w_0 by a factor k' (>1.0). This factor is a material property which increases with the speed of movement v of the edge of the contact (Maugis and Barquins, 1978; Greenwood and Johnson, 1981), i.e.

$$w_e' = k'(v)w_0.$$

Where the contact area is increasing, it is the action of surface forces which has to overcome the viscoelastic losses in addition to increasing the strain energy, so that the situation is reversed and the effective work of adhesion is reduced, i.e.

$$w_e'' = k''(v)w_0,$$

where $k''(v) < 1.0$ (Johnson, 1976). Experiments suggest that at typical velocities $k' \gg 1.0$ and $k'' \ll 1.0$.

We can now reconsider the previous loading and unloading cycle for such a material. The analysis leading to Fig. 3 still holds, but the value of α depends upon the *effective* work of adhesion w_e and is dependent upon the rate of change of contact dimension. During compression the effective work of adhesion is negligible ($k'' \ll 1.0$), so that α can be taken to be zero. The contact area increases with $\bar{p}$ according to the equilibrium curve; a pressure $\bar{p} = p^*$ is required to achieve full contact.

To initiate peeling from a flaw of size $2b_f$, the (tensile) value of p must exceed that given by eqn (15a), with $w = w_0$. When that is so, peeling will proceed under quasi-static (equilibrium) conditions. The rate of peeling will vary such that, at each value of a, the value of $\alpha(v)$ is such that equilibrium conditions are maintained.

4. TWO-DIMENSIONAL WAVINESS

The adhesionless contact of a flat surface with a surface which has orthogonal waves of equal amplitude and wavelength has been investigated by Johnson *et al.* (1985). The undeformed gap between the surfaces may be expressed:

$$h(x, y) = \Delta[1 - \cos(2\pi x/\lambda) \cdot \cos(2\pi y/\lambda)]. \tag{17}$$

The surfaces first touch at the crests of the waves, located at the points (0,0) and ($\lambda/2$,$\lambda/2$). The troughs are at points ($\lambda/2$,0) (0,$\lambda/2$); the mid-point ($\lambda/4$,$\lambda/4$) is a saddle point. If the mean pressure $\bar{p}$ exceeds p^*, where in this case $p^* = \sqrt{2}\pi E^*\Delta/\lambda$ contact is complete and the distribution of pressure is given by

$$p(x,y) = \bar{p} + p^* \cos(2\pi x/\lambda)\cos(2\pi y/\lambda). \tag{18}$$

When $\bar{p} < p^*$ contact is not complete and the problem requires numerical solution since the shape of the discrete contact areas changes during loading. At light loads the contacts close to the crests are circular and behave like independent Hertz contacts of spherical bodies. At the other extreme, when complete contact is approached, regions of no contact, located at the troughs, are also approximately circular. An asymptotic solution for this latter situation was obtained by Johnson *et al.* (1985), where the gap between the surfaces was found to be

$$g(\rho) = \frac{2p^*b}{\pi E^*}[(1-\bar{p}/p^*) - \tfrac{4}{9}(1+2\rho^2)](1-\rho^2)^{1/2}, \tag{19}$$

where b is the radius of the no contact area, $\rho = r/b$, $(\rho < 1.0)$ and r is the radial distance from the trough $(\lambda/2, \lambda/2)$. The value of b was determined by the condition that, in the absence of adhesion, the surfaces must separate smoothly at the edge of contact, i.e. $dg/d\rho = 0$ as $\rho \to 1.0$. For this condition to be satisfied

$$\frac{4}{3}\left(\frac{\pi b}{\lambda}\right)^2 = 1 - \frac{\bar{p}}{p^*}. \tag{20}$$

With adhesion a singularity in tension, corresponding to a stress intensity factor K_I would be expected at the edge of the contact. Its value is related to the asymptotic shape of the gap ("crack opening displacement") by

$$\frac{K_I}{(\pi b)} = \lim_{\rho \to 1}\left\{\frac{g(\rho)}{(1-\rho^2)^{1/2}}\right\}$$

$$= \frac{2p^*}{\pi}[(1-\bar{p}/p^*) - \tfrac{4}{3}(\pi b/\lambda)^2]. \tag{21}$$

Again, using the relationship $K_I^2/2E^* = w$, eqn (21) becomes

$$1 - \bar{p}/p^* = \frac{3}{3}\left(\frac{\pi b}{\lambda}\right)^2 + \frac{\pi}{2}\left(\frac{2E^*w}{\pi b p^{*2}}\right)^{1/2}, \tag{22}$$

which compares with eqn (20) for zero adhesion. For a small circular flaw of radius b_f $(\ll \lambda/2)$ located at the trough, we can write

$$\frac{\bar{p}_f}{p^*} = 1 - \frac{\pi}{2}\left(\frac{2E^*w}{\pi b_f p^{*2}}\right)^{1/2}. \tag{23}$$

This expression is equivalent to eqn (15b) for a one-dimensional wave. It prescribes the mean negative pressure $\bar{p}_f$ necessary to initiate peeling from a flaw of radius b_f in the manner described in the previous section.

In the case of two-dimensional waviness, small gaps between the surfaces located in a trough are isolated from the environment so that their behaviour could well be influenced in practice by trapped air. If air were trapped on loading it would be pressurised to such an extent that complete contact could not be achieved. This entrapment would then constitute a "flaw" which would promote peeling under tension, as prescribed by eqn (23).

Alternatively, a flaw which contains no gas will require a mean tensile stress which is enhanced by the value of the ambient atmospheric pressure in order to initiate separation.

Finally it has to be recognised that real surfaces are likely to have a fine scale roughness of a more random nature superimposed upon the waviness considered above. A distribution of asperity heights has the effect of reducing the overall adhesion energy by a process in which compressive forces between the higher asperites break the adhesive junctions between the lower ones. This process has been modelled by Fuller and Tabor (1975).

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